Singularities around $w = -1$

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Abstract. In this talk we would like to analyse the appearance of singularities in FLRW cosmological models which evolve close to $w = -1$, where w is the barotropic index of the universe. We relate small terms in cosmological time around $w = -1$ with the correspondent scale factor of the universe and check for the formation of singularities.

INTRODUCTION

Accelerated expansion of our universe has been established observationally in the last decade from several sources of information. Even more, the barotropic index w of the equation of state of the content of the universe, that is the ratio between pressure and energy density, has been checked to be close to -1, value which has been dubbed as the phantom divide. This value corresponds to a cosmological constant as energy content of the universe.

The fact of the accelerated expansion of the universe has lead to several scenarios for a singular fate of the universe, different from eternal expansion or Big Crunch. A classification of such singularities is provided in [1]. Among these new scenarios one can find Big Rip singularities [2], sudden singularities [3], Big Freeze singularities [4], w-singularities [5] or directional singularities [6].

Some of then however have been shown to be not strong enough [7] to mean the actual end of the universe. For instance, sudden singularities [8] and w-singularities [9] are weak singularities. They have also been studied within the framework of modified theories of gravitation [10].

The main idea behind the previous analysis is to assume that the scale factor of the universe may be written as generalised power expansion of the time coordinate [11] and then analyse the geodesic completeness [12] of the FLRW spacetime endowed with such scale factor.

With these ideas in mind, we would like to analyse the behaviour of cosmological models in the vicinity of $w = -1$. We shall show that there are scenarios which fall out of the previous classification and are to be regarded separately.

PERTURBED BAROTROPIC INDEX

Let us assume that the barotropic index behaves approximately as a cosmological constant at present time, that is,

$$w = -1 + h(t), \qquad |h(t)| \ll 1. \tag{1}$$

It is formally possible to get an explicit expression for the scale factor $a(t) = e^{f(t)}$ for such barotropic index,

$$w = \frac{p}{\rho} = -\frac{1}{3} - \frac{2}{3}\frac{a\ddot{a}}{\dot{a}^2} \Rightarrow f(t) = \frac{2}{3}\int \frac{dt}{\int h(t)\,dt + k} + C.$$

The constant C is irrelevant since it amounts to a change of scale or time origin, $a(t) \to e^C a(t)$. Since the density of the energy content is

$$\rho = \left(\frac{\dot{a}}{a}\right)^2 = \dot{f}^2 \Rightarrow \sqrt{\rho} = \frac{2}{3\int h + k},$$

we may use it to fix the constant k.

For instance, taking the origin in the near future and assuming a power-law deviation from unity, $h(t) = \alpha(-t)^p$, $p > 0$ so that $w(0) = -1$,

$$\sqrt{\rho(t)} = -\frac{2(p+1)}{3\alpha}\frac{1}{k + (-t)^{p+1}}.$$

Two different cases arise in these models:

- $k \neq 0$: $\sqrt{\rho}$ has simple poles out of $t = 0$ and $\rho \sim (t - t_0)^{-2}$, but $w = -1$ is regular.
- $k = 0$: Singular density and scale factor at $w = -1$. Naming $\beta = 2(1+p)/3\alpha p$,

$$\rho(t) = \frac{4}{9}\left(\frac{p+1}{\alpha}\right)^2 \frac{1}{t^{2p+2}}, \quad f(t) = -\frac{2}{3}\frac{1+p}{\alpha p}(-t)^{-p}, \qquad a(t) = e^{-\beta/(-t)^p}.$$

We are interested in the latter models, which have a singular density at the time when $w = -1$ is reached. These models have a non-analytical scale factor with an essential singularity at $t = 0$ and the energy density blow up as $1/t^{2p+2}$ instead of t^{-2}, which is the usual power for divergencies with analytical scale factors. Two possibilities arise depending on the sign of the constant α:

- For $\alpha > 0$, we have a *Great Crunch*: $a(t) \to 0$.
- For $\alpha < 0$, we have a *Great Rip*: $a(t) \to \infty$.

GEODESICS NEAR $w = -1$

Equations for causal geodesics parametrised by proper time τ,

$$d\tau = \sqrt{-g_{ij}dx^i dx^j},$$

in a flat FLRW model reduce to

$$\frac{dt}{d\tau}=\sqrt{\delta+\frac{P^2}{a^2(t)}},\qquad \frac{dr}{d\tau}=\pm\frac{P}{a^2(t)},$$

where P is a constant of motion and $\delta=0$ for null and $\delta=1$ for timelike geodesics.

The case of null geodesics is simpler since the equations may be integrated,

$$\tau=P^{-1}\int_{t_0}^{0}a(t)\,dt.$$

This means it takes an infinite proper time to Great Rip ($a(0)=\infty$), so this singularity is not accesible along lightlike geodesics.

On the contrary, for timelike geodesics,

$$\tau=\int_{t_0}^{0}\frac{dt}{\sqrt{1+P^2a^{-2}(t)}},$$

all geodesics reach $w=-1$ in finite proper time.

Hence $w=-1$ becomes singular but for null geodesics. This sort of behaviour is similar to the one in the vicinity of a Big Rip singularity.

STRONG SINGULARITIES

In spite of the singular character of $w=-1$, it could happen that the singularity could be not strong enough to be the end of the universe, since extended objects might avoid being disrupted by tidal forces on crossing the singularity.

This concept of strength of singularities was first coined by Ellis and Schmidt [12] and it was related to the cosmic censorship conjecture later on. It was further developed by other authors. For instance, for Tipler [13] a singularity is strong if the volume spanned by three Jacobi fields orthogonal to the velocity of the geodesic tends to zero as it approaches its end. Królak [14] suggests a less restrictive criterion, requiring just the derivative of the volume with respect to proper time to be negative.

Since these definitions involve calculations with Jacobi fields, checking the strength of singularities may be cumbersome. Fortunately, necessary and sufficient conditions [15] have been derived involving just integrals of some components of the curvature tensor along incomplete geodesics.

For instance, following Tipler's definition, a singularity is strong along a null geodesic of velocity u if and only if the integral

$$\int_0^{\tau}d\tau'\int_0^{\tau'}d\tau''R_{ij}u^iu^j$$

diverges as τ tends to τ_0, where τ_0 is the proper time assigned to the singularity.

For Królak's criterion the necessary and sufficient condition is the divergence of the integral

$$\int_0^{\tau}d\tau'R_{ij}u^iu^j.$$

as τ tends to τ_0.

Such calculations are simple in our case and allow us to conclude that singularities at $w = -1$ are strong according to both criteria. The same result is obtained for timelike geodesics.

CONCLUSIONS

We have shown that there are two possible behaviours for models with a barotropic index close to $w = -1$:

- Regular crossing of $w = -1$.
- Essential singularity at $w = -1$: Great Crunch / Rip.

For the latter models the energy density blows up at the singularity and its divergence is worse than t^{-2}. The essential singularities are strong, though null geodesics never reach the Great Rip. More details and references may be found elsewhere [16].

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